9

Evolutionary Dynamics

9.3 Properties of the Replicator System

Only the last part of the question might not be obvious. Let $p(t)$ be a trajectory of (9.5), and define

$$b(t) = \int_0^t \frac{dt}{a(p(t), t)},$$

which is possible since $a(p, t) > 0$. Clearly, $b(t)$ is positive and increasing. Let $q(t) = p(b(t))$. Then, by the Fundamental Theorem of the Calculus,

$$\begin{aligned}\dot{q}_i(t) &= \dot{b}(t)\dot{p}_i(b(t)) = \frac{1}{a(t)} a(t) p_i(b(t))(\pi_i(p(b(t))) - \bar{\pi}(p(b(t)))) \\ &= q_i(t)(\pi_i(q(t)) - \bar{\pi}(q(t))). \text{ Q.E.D.}\end{aligned}$$

9.13 The Dynamics of Rock-Paper-Scissors and Related Games

Let π_α, π_β, and π_γ be the payoffs to the three strategies. Then, we have

$$\begin{aligned}\pi_\alpha &= \beta r + (1 - \alpha - \beta)s = \beta(r - s) - \alpha s + s, \\ \pi_\beta &= \alpha s + (1 - \alpha - \beta)r = \alpha(s - r) - \beta r + r, \\ \pi_\gamma &= \alpha r + \beta s.\end{aligned}$$

It is easy to check that the average payoff is then

$$\begin{aligned}\bar{\pi} &= \alpha\pi_\alpha + \beta\pi_\beta + (1 - \alpha - \beta)\pi_\gamma \\ &= (r + s)(\alpha + \beta - \alpha^2 - \alpha\beta - \beta^2).\end{aligned}$$

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At any fixed point involving all three strategies with positive probability, we must have $\pi_\alpha = \pi_\beta = \pi_\gamma$. Solving these two equations, we find $\alpha = \beta = \gamma = 1/3$, which implies that $\bar{\pi} = (r + s)/3$.

In a replicator dynamic, we have

$$\dot{\alpha} = \alpha(\pi_\alpha - \bar{\pi}),$$
$$\dot{\beta} = \beta(\pi_\beta - \bar{\pi}).$$

Expanding these equations, we get

$$\dot{\alpha} = -2\alpha\beta s - (r + 2s)\alpha^2 + \alpha s + \alpha p(\alpha, \beta),$$
$$\dot{\beta} = -2\alpha\beta r - (2r + s)\beta^2 + \beta r + \beta p(\alpha, \beta),$$

where $p(\alpha, \beta) = (r + s)(\alpha^2 + \alpha\beta + \beta^2)$.

This is of course a nonlinear ordinary differential equation in two unknowns. It is easy to check that its unique fixed point for $\alpha, \beta > 0$ is $\alpha = \beta = 1/3$, the mixed strategy Nash equilibrium for this game.

For the dynamics, we linearize the pair of differential equations by evaluating the Jacobian matrix of the right-hand sides at the fixed point. The Jacobian is

$$J(\alpha, \beta) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$a_{11} = -2\beta s - 2\alpha(r + 2s) + s + p(\alpha, \beta) + \alpha(2\alpha + \beta)(r + s),$$
$$a_{12} = -2\alpha s + \alpha(\alpha + 2\beta)(r + s),$$
$$a_{21} = -2\beta r + \beta(2\alpha + \beta)(r + s),$$
$$a_{22} = r - 2\alpha r - 2\beta(2r + s) + p(\alpha, \beta) + \beta(\alpha + 2\beta)(r + s),$$

so

$$J(1/3, 1/3) = \frac{1}{3}\begin{pmatrix} -s & r - s \\ s - r & -r \end{pmatrix}.$$

The eigenvalues of the linearized system are thus

$$\frac{1}{6}\left[-(r + s) \pm i\sqrt{3}(r - s)\right].$$

We prove the assertions as follows:

a. The determinant of the Jacobian is $(r^2 - rs + s^2)/9$. This has a minimum where $2s - r = 0$, with the value $r^2/12 > 0$. This shows that the system is hyperbolic, and since the determinant is positive, it is a node or a focus.
b. The real parts of the eigenvalues are negative if and only if $r + s < 0$, and are positive if and only if $r + s > 0$.
c. The eigenvalues are complex for $r \neq s$.
d. If $r + s = 0$, the eigenvalues are purely imaginary, so origin is a center. We thus cannot tell how the nonlinear system behaves using the linearization.

However, we can show that the quantity $q(\alpha, \beta) = \alpha\beta(1 - \alpha - \beta)$ is constant along trajectories of the dynamical system. Assuming this (which we will prove in a moment), we argue as follows. Consider a ray R through the fixed point (1/3,1/3) pointing in the α-direction. Suppose $q(\alpha, \beta)$ is strictly decreasing along this ray (we will also prove this in a moment). Then, the trajectories of the dynamical system must be closed loops. To see this, note first that the fixed point cannot be a stable node, since if we start at a point on R near the fixed point, q decreases as we approach the fixed point, but q must be constant along trajectories, which is a contradiction. Thus, the trajectories of the system must be spirals or closed loops. But they cannot be spirals, because when they intersect R twice near the fixed point, the intersection points must be the same, since $q(\alpha, \beta)$ is constant on trajectories but decreasing on R near the fixed point.

To see that q is decreasing along R near (1/3,1/3), note that

$$q(1/3 + t, 1/3) = \frac{1}{3}\left(\frac{1}{3} - t\right)^2,$$

which has a derivative with respect to t that evaluates to $-2/9 < 0$ at $t = 0$.

To see that $q(\alpha, \beta)$ is constant along trajectories, note that the differential equations for the dynamical system, assuming $r = 1$, $s = -1$, can be written as

$$\dot{\alpha} = 2\alpha\beta + \alpha^2 - \alpha \tag{A9.1}$$
$$\dot{\beta} = -2\alpha\beta - \beta^2 + \beta. \tag{A9.2}$$

Then,

$$\frac{d}{dt}q(\alpha, \beta) = \frac{d}{dt}[\alpha\beta(1 - \alpha - \beta)]$$

$$= \beta(1-\alpha-\beta)\dot{\alpha} + \alpha(1-\alpha-\beta)\dot{\beta} + \alpha\beta(-\dot{\alpha}-\dot{\beta})$$
$$= 0,$$

where we get the last step by substituting the expressions for $\dot{\alpha}$ and $\dot{\beta}$ from (A9.1) and (A9.2).

9.14 The Lotka-Volterra Model and Biodiversity

a. This is simple algebra, though you should check that the restrictions on the signs of a, b, c,and d ensure that $p^* > 0$.

b. We have

$$\begin{aligned}
\frac{\dot{p}}{p} &= \frac{\dot{u}}{u} - \left[\frac{\dot{u}}{w} + \frac{\dot{v}}{w}\right] \\
&= \frac{\dot{u}}{u} - \left[p\frac{\dot{u}}{u} + (1-p)\frac{\dot{v}}{v}\right] \\
&= ap + b(1-p) - kw \\
&\quad - [p[ap + b(1-p) - kw] + (1-p)[cp + d(1-p) - kw]] \\
&= ap + b(1-p) - [p[ap + b(1-p)] + (1-p)[cp + d(1-p)]] \\
&= \pi_A - \bar{\pi}.
\end{aligned}$$

c. We have

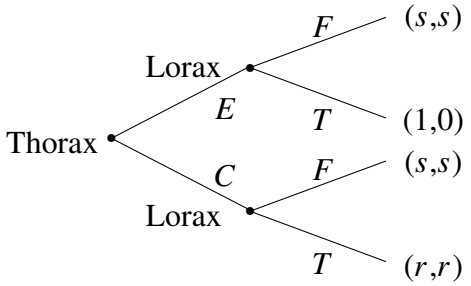
$$\begin{aligned}
\frac{\dot{p}_i}{p_i} &= \frac{\dot{u}_i}{u_i} - \sum_{j=1}^{n} \frac{\dot{u}_j}{u} \\
&= \frac{\dot{u}_i}{u_i} - \sum_{j=1}^{n} \frac{\dot{u}_j}{u_j} p_j \\
&= \sum_{j=1}^{n} a_{ij} p_j - kw - \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_{jk} p_k - ku\right) p_j \\
&= \sum_{j=1}^{n} a_{ij} p_j - ku - \sum_{j,k=1}^{n} a_{jk} p_k p_j + ku \sum_{k=1}^{n} p_j
\end{aligned}$$

$$= \sum_{j=1}^{n} a_{ij} p_j - \sum_{j,k=1}^{n} a_{jk} p_j p_k.$$

This proves the assertion, and the identification of the resulting equations as a replicator dynamic is clear from the derivation.

9.18 The Loraxes and Thoraxes

a. We have the following game tree, on which backward induction (pruning the game tree) gives the desired result.



b. It is easy to show that there is no pure strategy Nash equilibrium other than FE, so we look for a mixed strategy Nash equilibrium. Let p be the fraction of Thoraxes who cooperate. Then, the payoff to T is $pr + (1 - p) \cdot 0 = pr$, and the payoff to F is s, so in mixed strategy equilibrium we must have $pr = s$. Let q be the fraction of trusters among the Lorax. The payoff to C is then $qr + (1 - q)s$, and the payoff to E is $q + (1 - q)s$. Thus, Thoraxes always eat, so $p = 1$, which contradicts $pr = s$.

c. The payoff to I is $pr + (1 - p)s - \delta$ for a Lorax, and the payoff to NI is $\max\{pr, s\}$, since if $pr > s$, the noninspector chooses T, and with $pr < s$, the noninspector chooses F. Clearly, some Loraxes must be NI types, since if all were I types, then all Thoraxes would cooperate, so a Lorax could gain from trusting rather than inspecting. If both I and NI are used, they must have equal payoffs, so $p^* = 1 - \delta/s$ if $pr > s$, and $p^* = \delta/(r - s)$ if $pr < s$. Suppose $s/r > \delta/(r - s)$. Then, $p^* = 1 - \delta/s > 1 - (r - s)/r = s/r$ is consistent with $pr > s$. If $s/r < \delta/(r - s)$, then we must have $p^* = \delta/(r - s) > s/r$, which implies $p^* = 1 - \delta/s$. This is only possible if $\delta/(r - s) = 1 - \delta/s$, or $s/r = \delta/(r - s)$, which is a contradiction. Hence, there is no mixed

strategy Nash equilibrium in this case. Assuming $pr > s$, the payoff to C for a second-mover is r, and the payoff to E is $1 - q$, so a mixed strategy equilibrium requires $q^* = 1 - r$.

d. Suppose $p > s/r$, so NI implies T for a Lorax. Then, (payoff to I)−(payoff to NI) = $(1-p)s - \delta$. For a Thorax, (payoff to C)−(payoff to E) = $r - (1-q)$. Then, the linear dynamic satisfies the equations

$$\dot{p} = b(q - (1-r))$$
$$\dot{q} = -a\left[p - \left(1 - \frac{\delta}{s}\right)\right],$$

where $a, b > 0$ are rates of adjustment. The Jacobian at the fixed point $p^* = 1 - \delta/s$, $q^* = 1 - r$ is

$$J(p^*, q^*) = \begin{bmatrix} 0 & b \\ -a & 0 \end{bmatrix}.$$

This gives $\alpha = \text{Trace}(J(p^*, q^*))/2 = 0$, $\beta = \det(J(p^*, q^*)) = ab > 0$, $\gamma = \alpha^2 - \beta = -ab < 0$. Thus, the fixed point is a center.